

Phys 342: Homework #4
Due March 27

1. The goal of this problem is to understand how the fusion reaction rate depends on temperature. In class we showed that the reaction rate can be written as

$$r_{12} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_1 n_2}{(\pi\mu)^{1/2}} \int_0^\infty S(E) e^{-f(E)} dE \quad \text{where} \quad f(E) = \left(\frac{E_c}{E}\right)^{1/2} + \frac{E}{kT}$$

The function $e^{-f(E)}$ is sharply peaked near the Gamow peak E_0 . Assuming $S(E)$ is reasonably constant near the Gamow peak, we pull it out of the integral and write

$$r_{12} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_1 n_2}{(\pi\mu)^{1/2}} S(E_0) \int_0^\infty e^{-f(E)} dE$$

Our task is to estimate the remaining integral. Here are the steps to do that.

- (a) Expand the function $f(E)$ as a Taylor series around the Gamow peak. This means you can approximate f with the form

$$f \approx a_0 + a_1 (E - E_0) + a_2 (E - E_0)^2 + \dots$$

What I'm asking you to do is determine the coefficients a_0 , a_1 , and a_2 .

Hint. a_1 should be zero because the slope of f must be zero at the peak.

- (b) Using the approximate form for f , the integral becomes

$$\int_0^\infty e^{-f(E)} dE \approx e^{-a_0} \int e^{-a_2 x^2} dx$$

(I am not being very careful about the limits of integration, because as long as they are far from the peak they don't matter very much.) The integrand is now a Gaussian, and the integral of a Gaussian is simple:

$$\int e^{-a_2 x^2} dx = \left(\frac{\pi}{a_2}\right)^{1/2}$$

Use this with your values of a_0 and a_2 to write an approximation for r_{12} .

- (c) Now let's examine the temperature dependence of r_{12} . As discussed in class, if we approximate the temperature dependence as a power law, $r_{12} \propto T^\alpha$, we can find the power law index as follows:

$$\alpha = \frac{T}{r_{12}} \frac{dr_{12}}{dT}$$

Using your approximation for r_{12} , show that

$$\alpha = \left(\frac{E_c}{4kT} \right)^{1/3} - \frac{2}{3} \quad (1)$$

2. Now compute the temperature dependence of the reaction rate for two specific examples. If you did not complete the mathematical analysis above, you can just use eq. (1) for this problem. Take the temperature to be $T = 1.6 \times 10^7$ K.

- (a) PP chain: Consider the ${}^1_1\text{H} + {}^1_1\text{H}$ step in the chain. Compute kT , E_c , E_0 , and α for this reaction.
- (b) CNO cycle: Consider the ${}^{14}_7\text{N} + {}^1_1\text{H}$ step in the chain. Compute kT , E_c , E_0 , and α for this reaction.

3. On the wiki I placed a file with data from a detailed model of the Sun. You may use any appropriate software (Excel, Maple, etc.) to analyze the data and make plots, but please explain what you do. If you are unable to use such software, come talk to me.

- (a) Use the ideal gas law to compute and plot the average particle mass (in units of the proton mass) as a function of radius. Make sure to label the axes with appropriate units.

You should see three fairly distinct zones: (i) the inner 20% of the Sun; (ii) the region between 20–95% of the Sun’s radius; and (iii) the outer 5% of the Sun. Explain the general features you see in these three zones.

- (b) Plot the following two quantities

$$\left| \frac{dT}{dr} \right| \quad \text{and} \quad \frac{\gamma - 1}{\gamma} \frac{m}{k} g$$

as a function of radius. Show that the outer layer of the Sun is convective.

Hint. To take a derivative numerically, use

$$\frac{dT}{dr} \approx \frac{T(r + \Delta r) - T(r)}{\Delta r}$$

Recall that the acceleration due to gravity is

$$g(r) = \frac{GM(r)}{r^2}$$

4. (a) Use your knowledge of fusion in the Sun to estimate the number of solar neutrinos passing through your body each second.
- (b) The cross section for a neutrino to collide with a nucleus is around $\sigma \sim 10^{-44}$ cm². Estimate the number of times a neutrino will hit a nucleus in your body during your lifetime.